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ON THE LAGRANGIAN METHOD FOR STEADY AND UNSTEADY FLOW

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16. Abstract

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ON THE LAGRANGIAN METHOD FOR STEADY AND UNSTEADY FLOW

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ABSTRACT

A new and general Lagrangian formulation of fluid motion is given in which the independent variables are three material functions and a Lagrangian time, which differs for different fluid particles and is distinct from the Eulerian time. For steady flow it requires only three independent variables - the Lagrangian time and two stream functions - in contrast with the conventional Lagrangian formulation which apparently still requires four independent variables for describing a steady flow. This places the Lagrangian formulation for steady flow on the same footing as the Eulerian. For unsteady flow, the new formulation includes the conventional formulation as a special case when the Lagrangian time is identified with the Eulerian time and when the material functions are taken to be the fluid particle's position at some given time. The distinction between the Lagrangian and Eulerian time, however, is found useful in applications, e.g. to problems involving a free boundary.

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1. INTRODUCTION

It is well-known that there exist two general methods of description of fluid motion. The Eulerian method aims at describing the flow variables, e.g. velocity $\mathbf{v}$, as functions of the position $\mathbf{x} = (x,y,z)$ of a fluid particle at time t, viz., $\mathbf{v} = \mathbf{v}(x,y,z,t)$. In contrast, the Lagrangian method aims at describing the motion history of each fluid particle whose *name* is (a,b,c), viz., $\mathbf{x} = \mathbf{x}(a,b,c,t)$. Conventionally, the Lagrangian variables (a,b,c) of a particle are taken to be its position at some time $t = t_0$, say, i.e. $(a,b,c) = (x,y,z)|_{t=t_0}$.

The two methods are equivalent, except that the Lagrangian form gives more information: it tells where each fluid came from originally, this can considerably facilitate calculations of flows when two or more fluids, with different equations of state, are present. However, the Eulerian method has been the popular one and generally regarded as more advantageous than the Lagrangian for the following two reasons. Firstly, it directly leads to results showing explicitly the spatial distributions of flow quantities, such as the pressure field and the stress field, etc., this is desirable in some practical applications, e.g. flow over an airplane. Secondly, and more importantly, for steady flow the number of independent variables in the Eulerian method is immediately reduced by one - the time variable drops out. This is not easily done in the conventional Lagrangian method for which four independent variables are apparently still needed. This simplicity of the Eulerian method for steady flow over that of Lagrange explains why the former is used by the great majority of fluid dynamicists.

The situation is quite different for unsteady flow for which both methods require four independent variables and there seems no obvious advantage of the Eulerian method over the Lagrangian. In fact, while there is no distinction between steady and unsteady flow in the Lagrangian concept, the Eulerian method treats the steady flow as a degenerated case (of unsteady flow). As such, any Eulerian approach that is specially designed for steady problems may not readily be generalized to unsteady problems. Indeed, it has been demonstrated by Van Dommelen and Shen (1980) for unsteady boundary layer separation and by Hui and Tobak (1981) and Hui and Van Roessel (1984) for unsteady high Mach number flow that the Lagrangian method can be more advantageous than the Eulerian in treating unsteady flow problems.

The purpose of this paper is to develop a new and general Lagrangian method which uses as Lagrangian variables three material functions and a Lagrangian time τ distinct from the Eulerian time t. It will be shown that while the new Lagrangian

method includes the conventional one as a special case, it requires only three independent variables in describing a three-dimensional steady flow: the Lagrangian time and two material functions which for steady flow are stream functions. This then places the Lagrangian method on the same footing as the Eulerian one. The advantage of the new Lagrangian formulation over the conventional one for unsteady flow will also be discussed.

2. THE MATERIAL FUNCTIONS

2.1 Governing Equations in Eulerian Form

We start from Eulerian formulation to derive our new Lagrangian formulation. Consider a typical problem of fluid flow resulting from a rigid body moving through it. Let the motion of the body be prescribed by the velocity $\mathbf{v}_c(t)$, of its centre of mass and its angular velocity $\boldsymbol{\Omega}(t)$, and let the fluid be compressible, viscous, homogeneous, and isotropic. In a body-fixed coordinate system, the equations expressing conservation of mass, momentum and energy are the continuity equation, the Navier-Stokes equation, and the energy equation respectively:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0, \tag{1}$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \frac{1}{\rho}\nabla p = \mathbf{F} + \frac{\mu+\tilde{\mu}}{\rho}\nabla(\nabla\cdot\mathbf{v}) + \frac{\mu}{\rho}\nabla^2\mathbf{v}, \tag{2}$$

$$\rho c_v\left[\frac{\partial T}{\partial t} + \mathbf{v}\cdot\nabla T\right] + p\,\nabla\cdot\mathbf{v} = \nabla\cdot(k\nabla T) + (\tilde{\mu} + \frac{2}{3}\mu)(\nabla\cdot\mathbf{v})^2, \tag{3}$$

with the equation of state

$$p = \rho R T, \tag{4}$$

where p is pressure, ρ density, T temperature, μ the shear viscosity, $\tilde{\mu}$ the bulk viscosity, k the termal conductivity, c_p and c_v the specific heats of the fluid, and $R = c_p - c_v$. $\mathbf{F}$ in Equation (2) is the sum of the body force per unit mass $\mathbf{G}$ and the inertia force arising from the motion of the body,

$$\mathbf{F} = \mathbf{G} - \left[\frac{d\mathbf{v}_c}{dt} + 2\boldsymbol{\Omega}\times\mathbf{v} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{x}) + \frac{d\boldsymbol{\Omega}}{dt}\times\mathbf{x}\right] \tag{5}$$

Let $\xi^i (i = 1,2,3)$ be body-fixed curvilinear coordinates such that $\xi^1 = 0$, say, coincides with the body surface. Then equations (1) to (3) may be written, with the use of

summation convention,

$$\frac{\partial \rho}{\partial t} + \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi^i} \left(\sqrt{g}\, \rho v^i \right) = 0, \tag{6}$$

$$\begin{aligned}
\frac{\partial v^i}{\partial t} + v^j \frac{\partial v^i}{\partial \xi^j} + \Gamma^i_{jk} v^j v^k + \frac{1}{\rho} g^{ij} \frac{\partial p}{\partial \xi^j} &= F^i + \frac{\mu + \tilde{\mu}}{\rho} g^{ij} \frac{\partial \theta}{\partial \xi^j} \\
&+ \frac{\mu}{\rho} g^{jk} \left\{ \frac{\partial^2 v^i}{\partial \xi^j \partial \xi^k} + \Gamma^i_{kl} \frac{\partial v^l}{\partial \xi^j} + \Gamma^i_{jl} \frac{\partial v^l}{\partial \xi^k} - \Gamma^l_{jk} \frac{\partial v^i}{\partial \xi^l} \right. \\
&\left. + \left(\Gamma^i_{jl,k} + \Gamma^i_{mk} \Gamma^m_{lj} - \Gamma^i_{lm} \Gamma^m_{jk} \right) v^l \right\},
\end{aligned} \tag{7}$$

$$\rho c_v \left(\frac{\partial T}{\partial t} + v^j \frac{\partial T}{\partial \xi^j} \right) + p\theta = \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi^j} \left(\sqrt{g}\, k g^{jk} \frac{\partial T}{\partial \xi^k} \right) + \left(\tilde{\mu} + \frac{2}{3} \mu \right) \theta^2, \tag{8}$$

where the g^{ij}'s are the components of the inverse to the metric, g is the determinant of the metric, the Γ^i_{jk}'s are the Christoffel symbols of the second kind, and

$$\theta = \nabla \cdot \mathbf{v} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi^j} \left(\sqrt{g}\, v^j \right). \tag{9}$$

In the special case of inviscid flow of a perfect gas the equations of motion may be obtained from (6) and (7) by putting $\mu = \tilde{\mu} = 0$, and by replacing (8) by

$$\frac{\partial}{\partial t} \left(\frac{p}{\rho^\gamma} \right) + v^j \frac{\partial}{\partial \xi^j} \left(\frac{p}{\rho^\gamma} \right) = 0 \tag{10}$$

where $\gamma = c_p/c_v$.

2.2 The Material Functions

The key to the development of the new Lagrangian formulation lies in the handling of the continuity equation.

It has long been known that for steady two-dimensional or axisymmetric flows a stream function may be used to eliminate the continuity equation and reduce the number of unknown functions by one. It was Lagrange (1781) who first proposed the stream function for the two-dimensional motion of an incompressible fluid and sixty-one years later Stokes (1842) used a stream function for axisymmetric flows of an incompressible fluid in precisely the same manner. Since then the use of stream functions has become a standard technique for solving two-dimensional or axisymmetric flows whether compressible or incompressible. The use of two stream functions for

three-dimensional flows, while less well known, has been considered by a number of people, among them Yih (1957).

For unsteady flows the presence of the $\frac{\partial \rho}{\partial t}$ term in the continuity equation (1) however, precludes the use of stream functions. But, facilitated by a change of notation, it is possible to introduce a set of new functions to be called *material functions*.

Thus letting $\xi^0 = t$, and noting that the metric is time independent, we may re-write the continuity equation as

$$\frac{\partial}{\partial \xi^I}(\sqrt{g}\,\rho v^I) = 0, \tag{11}$$

where $v^0 = 1$ and upper case lating indices range from 0 to 3. We now introduce three functions $\psi^i(\xi^0,\xi^1,\xi^2,\xi^3)$, $i = 1,2,3$, related to the components of velocity as follows

$$\sqrt{g}\,\rho v^I = \epsilon^{IJKL}\lambda(\psi^1,\psi^2,\psi^3)\frac{\partial \psi^1}{\partial \xi^J}\frac{\partial \psi^2}{\partial \xi^K}\frac{\partial \psi^3}{\partial \xi^L}, \quad (I = 0,1,2,3) \tag{12}$$

where λ is an arbitrary function of the ψ^i's. Substitution into (11) shows that the continuity equation is identically satisfied. Taking the material derivative of ψ^i yields

$$\begin{aligned}
\frac{D\psi^i}{Dt} &= \frac{\partial \psi^i}{\partial t} + v^j\frac{\partial \psi^i}{\partial \xi^j} = v^I\frac{\partial \psi^i}{\partial \xi^I} \\
&= \frac{1}{\sqrt{g}\,\rho}\epsilon^{IJKL}\lambda(\psi^i)\frac{\partial \psi^i}{\partial \xi^I}\frac{\partial \psi^1}{\partial \xi^J}\frac{\partial \psi^2}{\partial \xi^K}\frac{\partial \psi^3}{\partial \xi^L} \\
&= 0. \qquad\qquad (i = 1,2,3)
\end{aligned} \tag{13}$$

This shows that the ψ^i's are constant following the fluid and so the name material functions is appropriate. The existence of these functions was known by Yih (1957) but they have been little utilized since then.

The material functions introduced above for unsteady flow are a generalization of the stream functions for steady flow. In the latter case the time variable $t(= \xi^0)$ does not appear, consequently only two material functions $\psi^i(\xi^1,\xi^2,\xi^3)$, $i = 1,2$, are needed to eliminate the continuity equation, and (12) reduce to

$$\sqrt{g}\,\rho v^i = \epsilon^{ijk}\lambda(\psi^1,\psi^2)\frac{\partial \psi^1}{\partial \xi^j}\frac{\partial \psi^2}{\partial \xi^k}, \quad (i = 1,2,3) \tag{14}$$

which show that the material functions in steady flow are just the well-known stream functions.

Since ψ^i = const, (i = 1,2,3) following a fluid particle, a particle can, in turn, be identified by a set of the constants ψ^i. In other words, the material functions ψ^i can be and will be in this paper, used as Lagrangian variables. We also note that the material functions, analogues to the stream functions in steady flow, are not uniquely determined by (12) but, rather, there are freedoms in choosing them. The conventional choice of the Lagrangian variable (a,b,c) as the particle's position at some time t_0 is just a particular choice of the material functions. In this regard, the conventional Lagrangian formulation is a special case of the present one.

It should also be noted that although Yih (1957) pointed out that the arbitrary function $\lambda(\psi^i)$ may be taken to be unity without loss of generality, it is not always desirable to do so in practical applications as illustrated by an example in section 6.

The body surface, which must be a material surface, can be described by $\psi^1 = 0$, say.

3. THE LAGRANGIAN TIME

We now further introduce a function $\psi^0(\xi^0,\xi^1,\xi^2,\xi^3)$ such that

$$\frac{\partial(\psi^0,\psi^1,\psi^2,\psi^3)}{\partial(\xi^0,\xi^1,\xi^2,\xi^3,)} = \frac{\sqrt{g}\,\rho}{\lambda(\psi^1,\psi^2,\psi^3)} = \frac{1}{J}. \tag{15}$$

When $\xi^I (I = 0,1,2,3)$ are regarded as functions of $\psi^J (J = 0,1,2,3)$, it can easily be shown that

$$\frac{\partial \xi^I}{\partial \psi^0} = v^I, \tag{16}$$

i.e.

$$\frac{\partial t}{\partial \psi^0} = 1 \tag{17}$$

and

$$\frac{\partial \xi^i}{\partial \psi^0} = v^i \quad (i = 1,2,3). \tag{18}$$

Also, the material derivative reduces to

$$\frac{D}{Dt} = \frac{\partial}{\partial \psi^0}, \tag{19}$$

which shows that ψ^0 is a measure of time following a fluid particle or *Lagrangian time.*

For this reason, ψ^0 will be replaced by τ whenever convenient. From now on we shall use $\psi^I (I = 0,1,2,3)$ as our Lagrangian variables.

The Lagrangian time τ differs from the Eulerian time t by an arbitrary function of ψ^i.

$$t = \tau + s(\psi^1,\psi^2,\psi^3), \tag{20}$$

as easily seen from Equation (17). This arbitrary function s may be taken to be zero, thereby identifying the two times in the unsteady flow case. However, one can also take advantage of this new freedom in applications. For instance, in the case of inviscid supersonic flow with shock wave past a body, one can choose $s(\psi^i)$ to be such that the unknown shock wave position is given by $\tau = 0$. This amounts to fixing the zero time of a particle at the moment when it crosses the shock. As shown by Van Roessel and Hui (1986) this renders the free boundary problem of flow with shock wave in the Eulerian $\{\xi^I\}$ space to a fixed boundary one in the Lagrangian $\{\psi^I\}$ space, the latter being easier to solve.

In the case of steady flow, while the Eulerian time t does not appear, the Lagrangian time $\tau(=\psi^0)$ still does and is defined by

$$\frac{\partial(\psi^0,\psi^1,\psi^2)}{\partial(\xi^1,\xi^2,\xi^3)} = \frac{\sqrt{g}\,\rho}{\lambda(\psi^1,\psi^2,\psi^3)}, \tag{21}$$

where the two stream functions ψ^1 and ψ^2 are defined by (14). Together with these stream functions, $\{\tau,\psi^1,\psi^2\}$ form a set of three independent Lagrangian variables for describing a three-dimensional steady flow. Here τ plays the dual role of being the time variable and of distinguishing fluid particles.

The Lagrangian time is thus seen to be more fundamental than the Eulerian time when both steady and unsteady flow are considered.

4. LAGRANGIAN FORMULATION

Making the transformation of independent variables from $\{\xi^I\}$ to $\{\psi^I\}$ in (6) to (8), one obtains the following equations of motion in Lagrangian form

$$\frac{\partial^2\xi^i}{\partial\tau^2} + \Gamma^i_{jk}\frac{\partial\xi^j}{\partial\tau}\frac{\partial\xi^k}{\partial\tau} + \frac{1}{\rho}g^{ij}N_j[p] = F^i + \frac{\mu+\tilde{\mu}}{\rho}g^{ij}N_j[\theta]$$

$$+\frac{\mu}{\rho}g^{jk}\left\{N_jN_k[\frac{\partial\xi^i}{\partial\tau}]+\Gamma^i_{kl}N_j[\frac{\partial\xi^l}{\partial\tau}]+\Gamma^i_{jl}N_k[\frac{\partial\xi^l}{\partial\tau}]-\Gamma^l_{jk}N_l[\frac{\partial\xi^i}{\partial\tau}]\right.$$

$$\left.+\left(N_k[\Gamma^i_{jl}]+\Gamma^i_{mk}\Gamma^m_{lj}-\Gamma^i_{lm}\Gamma^m_{jk}\right)\frac{\partial\xi^l}{\partial\tau}\right\},\ (i=1,2,3) \tag{22}$$

$$\rho c_v\frac{\partial T}{\partial\tau}+p\theta=\frac{1}{\sqrt{g}}N_j[\sqrt{g}kg^{jk}N_k[T]]+(\tilde{\mu}+\frac{2}{3}\mu)\theta^2, \tag{23}$$

$$\frac{\partial(t,\xi^1,\xi^2,\xi^3)}{\partial(\tau,\psi^1,\psi^2,\psi^3)}=\frac{\lambda(\psi^1,\psi^2,\psi^3)}{\sqrt{g}\rho} \tag{24}$$

$$t=\tau+s(\psi^1,\psi^2,\psi^3) \tag{25}$$

where

$$\theta=\frac{1}{\sqrt{g}}N_j[\sqrt{g}\frac{\partial\xi^j}{\partial\tau}] \tag{26}$$

and the operator N_I $(I = 0,1,2,3)$ is

$$\begin{aligned}N_I&=\frac{\partial}{\partial\xi^I}\\&=\frac{\sqrt{g}\rho}{\lambda}e_{IJKL}\left\{\frac{\partial\xi^J}{\partial\psi^1}\frac{\partial\xi^K}{\partial\psi^2}\frac{\partial\xi^L}{\partial\psi^3}\frac{\partial}{\partial\tau}-\frac{\partial\xi^J}{\partial\psi^2}\frac{\partial\xi^K}{\partial\psi^3}\frac{\partial\xi^L}{\partial\tau}\frac{\partial}{\partial\psi^1}\right.\\&\qquad\left.+\frac{\partial\xi^J}{\partial\psi^3}\frac{\partial\xi^K}{\partial\tau}\frac{\partial\xi^L}{\partial\psi^1}\frac{\partial}{\partial\psi^2}-\frac{\partial\xi^J}{\partial\tau}\frac{\partial\xi^K}{\partial\psi^1}\frac{\partial\xi^L}{\partial\psi^2}\frac{\partial}{\partial\psi^3}\right\}.\end{aligned} \tag{27}$$

Equation (22) to (25) are six equations for six unknowns p,ρ,ξ^I $(I = 0,1,2,3)$. Of course, Equation (25) can be used to eliminate t and (24) to eliminate ρ, resulting in four equations for four unknowns.

Two-dimensional and one-dimensional flow equations in Lagrangian form are given in the Appendix.

5. SPECIAL CASES

5.1 Inviscid Flow

In the case of inviscid flow of a perfect gas Equations (22) are simplified by putting $\mu = \tilde{\mu} = 0$, while (23) is replaced by

$$\frac{\partial}{\partial r}\left(\frac{p}{\rho^{\gamma}}\right) = 0 \tag{28}$$

which is readily integrated.

5.2 Steady Flow

For steady flow, the two stream functions ψ^1 and ψ^2 and $\lambda(\psi^1,\psi^2)$ are defined by (14), whereas the Lagrangian time r is defined by (21). The governing equations for steady flow in Lagrangian form can be derived by a method (Van Roessel, 1985) similar to that leading to Equations (22) - (25) for unsteady flow. Alternatively, and which amounts to the same, the steady flow equations can be obtained as a special case of the unsteady flow equations by contracting both Eulerian time t and the third material functions ψ^3 to the Lagrangian time r. Thus Equation (24) reduces to

$$\frac{\partial(\xi^1,\xi^2,\xi^3)}{\partial(r,\psi^1,\psi^2)} = \frac{\lambda(\psi^1,\psi^2)}{\sqrt{g}\,\rho}, \tag{29}$$

while Equation (25) becomes redundant. On the other hand, Equations (22) and (23) remain unchanged in form except that the operator N_i becomes

$$N_i = \frac{\sqrt{g}\,\rho}{\lambda} e_{ijk}\left\{\frac{\partial\xi^j}{\partial\psi^1}\frac{\partial\xi^k}{\partial\psi^2}\frac{\partial}{\partial r} + \frac{\partial\xi^j}{\partial\psi^2}\frac{\partial\xi^k}{\partial r}\frac{\partial}{\partial\psi^1} + \frac{\partial\xi^j}{\partial r}\frac{\partial\xi^k}{\partial\psi^1}\frac{\partial}{\partial\psi^2}\right\}. \tag{30}$$

5.3 Cartesian Coordinates

In cartesian coordinates $\xi^i = (x,y,z)$, $\Gamma^i_{jk} = 0$ $g^{ij} = \delta^{ij}$, $g = 1$, Equation (22) and (23) reduce to

$$\frac{\partial^2\xi^i}{\partial r^2} + \frac{1}{\rho}N_i[p] = F^i + \frac{\mu+\tilde{\mu}}{\rho}N_i[\theta] + \frac{\mu}{\rho}N_jN_j[\frac{\partial\xi^i}{\partial r}], \quad (i = 1,2,3) \tag{31}$$

$$\rho c_v\frac{\partial T}{\partial r} + p\theta = N_j[kN_j[T]] + (\tilde{\mu} + \frac{2}{3}\mu)\theta^2, \tag{32}$$

whereas (24) to (27) are unchanged in form except that $g = 1$.

6. AN EXAMPLE: GERSTNER WAVE

Consider two-dimensional, inviscid, incompressible, flow of water under gravity $\bar{g}$. Use cartesian coordinates xy, with x horizontal and y vertically upward. Let $t = \tau$ and $\psi^1 = \alpha$, $\psi^2 = \beta$. The governing equations are

$$\frac{\partial(x,y)}{\partial(\alpha,\beta)} = \frac{\lambda(\alpha,\beta)}{\rho},$$

$$\frac{\partial^2 x}{\partial \tau^2} + \frac{1}{\lambda}\frac{\partial(p,y)}{\partial(\alpha,\beta)} = 0,$$

$$\frac{\partial^2 y}{\partial \tau^2} + \frac{1}{\lambda}\frac{\partial(x,p)}{\partial(\alpha,\beta)} = -\bar{g}.$$

It is known that this system admits an exact analytic solution representing travelling waves (Gerstner, 1802) as follows

$$x = \alpha - a e^{k\beta}\sin(k\alpha - \omega\tau),$$

$$y = \beta + a e^{k\beta}\cos(k\alpha - \omega\tau),$$

$$p = -\rho\bar{g}\beta + \frac{1}{2}\rho a^2\omega^2(e^{2k\beta} - 1),$$

$$\lambda = \rho(1 - a^2 k^2 e^{2k\beta}),$$

where a and k are arbitrary constants, and $\omega^2 = \bar{g}k$.

It is noted that $\lambda = \lambda(\beta)$ in this case. If $\lambda = const$ was imposed, the same Gerstner wave solution would be extremely difficult to obtain.

7. CONCLUSIONS

Starting from the Eulerian formulation, a new and general Lagrangian formulation for fluid flow is developed. This new formulation uses as Lagrangian variables three material functions and a Lagrangian time distinct from the Eulerian time.

Whilst the Eulerian time is the same for all fluid particles, the Lagrangian time differs for different particles. It thus plays a dual role of being the time variable and simultaneously of distinguishing fluid particles. With the use of Lagrangian time, steady flow can be described by three independent variables - the Lagrangian time and two stream functions, thus placing the Lagrangian formulation for steady flow on the same footing as Eulerian. This is in direct contrast with the conventional Lagrangian method which apparently still requires four independent variables for describing a

steady flow - the Eulerian time plus three variables to identify a particle in three-dimensional space.

For unsteady flow the new formulation includes the conventional formulation as a special case when the Lagrangian time is identified with the Eulerian time and when the material functions are taken to be the particle position at some given time. However, the distinction between Lagrangian and Eulerian time may prove useful in practical applications.

Finally, the present formulation allows simultaneous presence of both Lagrangian and Eulerian variables, thus facilitating the change from the former to the latter to give results showing the explicit spatial distribution of flow quantities. An example of this is given by Van Roessel and Hui (1980).

APPENDIX: LOWER-DIMENSIONAL FORMULATION

For two-dimensional unsteady flow, Equations (22) - (25) become

$$\frac{\partial^2 \xi^\alpha}{\partial \tau^2} + \Gamma^\alpha_{\beta\gamma} \frac{\partial \xi^\beta}{\partial \tau} \frac{\partial \xi^\gamma}{\partial \tau} + \frac{1}{\rho} g^{\alpha\beta} N_\beta[p] = F^\alpha + \frac{\mu + \tilde{\mu}}{\rho} g^{\alpha\beta} N_\beta[\theta]$$

$$+ \frac{\mu}{\rho} g^{\beta\gamma} \Big\{ N_\beta N_\gamma [\frac{\partial \xi^\alpha}{\partial \tau}] + \Gamma^\alpha_{\gamma\nu} N_\beta [\frac{\partial \xi^\nu}{\partial \tau}] + \Gamma^\alpha_{\beta\nu} N_\gamma [\frac{\partial \xi^\nu}{\partial \tau}] - \Gamma^\nu_{\beta\gamma} N_\nu [\frac{\partial \xi^\alpha}{\partial \tau}]$$

$$+ \left(N_\gamma [\Gamma^\alpha_{\beta\mu}] + \Gamma^\alpha_{\lambda\gamma} \Gamma^\lambda_{\mu\beta} - \Gamma^\alpha_{\mu\lambda} \Gamma^\lambda_{\beta\gamma} \right) \frac{\partial \xi^\mu}{\partial \tau} \Big\}, \quad (\alpha = 1,2) \tag{A1}$$

$$\rho c_v \frac{\partial T}{\partial \tau} + p\theta = \frac{1}{\sqrt{g}} N_\alpha [\sqrt{g} k g^{\alpha\beta} N_\beta [T]] + \left(\tilde{\mu} + \frac{2}{3}\mu \right) \theta^2, \tag{A2}$$

$$\frac{\partial (t, \xi^1, \xi^2)}{\partial (\tau, \psi^1, \psi^2)} = \frac{\lambda(\psi^1, \psi^2)}{\sqrt{g}\,\rho}, \tag{A3}$$

$$t = \tau + s(\psi^1, \psi^2), \tag{A4}$$

where

$$N_{i'} = \frac{\sqrt{g}\,\rho}{\lambda} e_{i'j'k'} \left\{ \frac{\partial \xi^{j'}}{\partial \psi^1} \frac{\partial \xi^{k'}}{\partial \psi^2} \frac{\partial}{\partial \tau} + \frac{\partial \xi^{j'}}{\partial \psi^2} \frac{\partial \xi^{k'}}{\partial \tau} \frac{\partial}{\partial \psi^1} + \frac{\partial \xi^{j'}}{\partial \tau} \frac{\partial \xi^{k'}}{\partial \psi^1} \frac{\partial}{\partial \psi^2} \right\}, \tag{A5}$$

$$(i' = 0,1,2)$$

$$\theta = \frac{1}{\sqrt{g}} N_\alpha [\sqrt{g} \frac{\partial \xi^\alpha}{\partial \tau}]. \tag{A6}$$

Similarly, for one-dimensional unsteady flow, $\Gamma^1_{11} = \frac{g'}{2g}$, $g^{11} = \frac{1}{g}$, $F^1 = F$, $\xi^1 = \xi$, $\psi^1 = \psi$, $N_1 = N$, and the governing equations reduce to

$$\frac{\partial^2 \xi}{\partial \tau^2} + \frac{g'}{2g} \left(\frac{\partial \xi}{\partial \tau} \right)^2 + \frac{1}{\rho g} N[p] = F + \frac{\mu + \tilde{\mu}}{\rho g} N[\theta]$$

$$+ \frac{\mu}{\rho g} \left(N^2 [\frac{\partial \xi}{\partial \tau}] + N [\frac{g'}{2g} \frac{\partial \xi}{\partial \tau}] \right), \tag{A7}$$

$$\rho c_v \frac{\partial T}{\partial \tau} + p\theta = \frac{1}{\sqrt{g}} N [\frac{k}{\sqrt{g}} N[T]] + (\tilde{\mu} + \frac{2}{3}\mu) \theta^2, \tag{A8}$$

$$\frac{\partial(t,\xi)}{\partial(\tau,\psi)} = \frac{\lambda(\psi)}{\sqrt{g}\,\rho} \tag{A9}$$

$$t = \tau + s(\psi) \tag{A10}$$

$$N = \frac{\sqrt{g}\,\rho}{\lambda}\left(\frac{\partial t}{\partial t}\;\frac{\partial}{\partial\psi} - \frac{\partial t}{\partial\psi}\;\frac{\partial}{\partial r}\right), \tag{A11}$$

$$0 = \frac{1}{\sqrt{g}}\,N[\sqrt{g}\,\frac{\partial\xi}{\partial r}] \tag{A12}$$

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15. Supplementary Notes

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16. Abstract

ABS

A new and general Lagrangian formulation of fluid motion is given in which the independent variables are three material functions and a Lagrangian time, which differs for different fluid particles and is distinct from the Eulerian time. For steady flow it requires only three independent variables - the Lagrangian time and two stream functions - in contrast with the conventional Lagrangian formulation which apparently still requires four independent variables for describing a steady flow. This places the Lagrangian formulation for steady flow on the same footing as the Eulerian. For unsteady flow, the new formulation includes the conventional formulation as a special case when the Lagrangian time is identified with the Eulerian time and when the material functions are taken to be the fluid particle's position at some given time. The distinction between the Lagrangian and Eulerian time, however, is found useful in applications, e.g. to problems involving a free boundary.

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